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A Categorical Approach to T_1 Separation and the Product of State Property Systems[†]

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The category **SP** of state property systems and their morphisms was presented by Aerts *et al.* In the present article I will present the functors which establish the equivalence of this category **SP** and the category **Cls** of closure spaces and continuous maps. Aerts *et al.* and Van Steirteghem proved that T_0 closure spaces correspond to 'state determined' state property systems. In this paper I will show that T_1 closure spaces correspond to 'atomistic' state property systems. I also use the equivalence between the categories **SP** and **Cls** to construct the product of state property systems.

1. THE EQUIVALENCE OF THE CATEGORIES SP AND Cls

In this section I will show the equivalence of the category of the state property systems and the category of the closure spaces. Before associating with each state property system a closure space, I recall the definitions of these two concepts. For the proofs of results cited in this section I refer to ref. 2 and for the necessary category theory I refer to refs. 1 and 7.

Definition 1 (State property system, Cartan map). A triple $(\Sigma, \mathscr{L}, \xi)$ is called a state property system if Σ is a set, $\mathscr{L}$ is a complete lattice, and $\xi: \Sigma \to \mathscr{P}(\mathscr{L})$ is a function such that for $p \in \Sigma$, 0 the minimal element of $\mathscr{L}$, and $(a_i)_i \in \mathscr{L}$, we have

$$0 \notin \xi(p) \tag{1}$$

$$a_i \in \xi(p) \quad \forall i \Rightarrow \wedge_i a_i \in \xi(p) \tag{2}$$

and for $a, b \in \mathscr{L}$ we have

[†]This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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$$a < b \Leftrightarrow \forall r \in \Sigma: \quad a \in \xi(r) \quad \text{then} \quad b \in \xi(r) \tag{3}$$

If $(\Sigma,\mathscr{L},\xi)$ and $(\Sigma',\mathscr{L}',\xi')$ are state property systems, then

$$(m, n): (\Sigma',\mathscr{L}',\xi') \to (\Sigma,\mathscr{L},\xi)$$

is called an **SP**-morphism if $m: \Sigma' \to \Sigma$ and $n: \mathscr{L} \to \mathscr{L}'$ are functions such that for $a \in \mathscr{L}$ and $p' \in \Sigma'$

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \tag{4}$$

The category of state property systems and their morphisms is denoted by **SP**. If $(\Sigma,\mathscr{L},\xi)$ is a state property system, then its Cartan map is the mapping $\kappa: \mathscr{L} \to \mathscr{P}(\Sigma)$ defined by

$$\kappa: \quad \mathscr{L} \to \mathscr{P}(\Sigma): \qquad a \mapsto \kappa(a) = \{p \in \Sigma | a \in \xi(p)\} \tag{5}$$

For each state property system $(\Sigma,\mathscr{L},\xi)$ we can define a preorder on Σ as follows: For $p,q \in \Sigma$, we put

$$p < q \Leftrightarrow \xi(q) \subset \xi(p) \tag{6}$$

The physical interpretation of this mathmatical structure is the following. Considering an entity S, the set Σ consists of states of S, while the set $\mathscr{L}$ consists of properties of S. These two sets are linked by means of a function $\xi: \Sigma \to \mathscr{P}(\mathscr{L})$ which maps a state p to the set $\xi(p)$ of all properties that are actual in state p. We also say that a state p makes the property a actual iff $a \in \xi(p)$.

Definition 2 (Closure space). A closure space $(X, \mathscr{F})$ consists of a set X and a family of subsets $\mathscr{F} \subset \mathscr{P}(X)$ satisfying the following conditions:

$$\emptyset \in \mathscr{F} \tag{7}$$

$$(F_i)_i \in \mathscr{F} \Rightarrow \cap_i F_i \in \mathscr{F} \tag{8}$$

The closure operator corresponding to the closure space $(X, \mathscr{F})$ is defined as

$$cl: \quad \mathscr{P}(X) \to \mathscr{P}(X): \quad A \mapsto \cap \{F \in \mathscr{F} | A \subset F\} \tag{9}$$

If $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ are closure spaces, then a function $f: (X, \mathscr{F}) \to (Y, \mathscr{G})$ is called a continuous map if $\forall B \in \mathscr{G}: f^{-1}(B) \in \mathscr{F}$. The category of closure spaces and continuous maps is denoted by **Cls**.

The following theorem shows how we can associate with each state property system a closure space and with each morphism a continuous map.

Theorem 1. The correspondence F: **SP** $\to$ **Cls** consisting of
(1) the mapping

$$|\mathbf{SP}| \to |\mathbf{Cls}| \tag{10}$$

$$(\Sigma, \mathscr{L}, \xi) \mapsto F(\Sigma, \mathscr{L}, \xi) = (\Sigma, \kappa(\mathscr{L})) \tag{11}$$

(2) for every pair of objects $(\Sigma, \mathscr{L}, \xi)$, $(\Sigma', \mathscr{L}', \xi')$ of **SP** the mapping

$$\mathbf{SP}((\Sigma', \mathscr{L}', \xi'), (\Sigma, \mathscr{L}, \xi)) \to \mathbf{Cls}(F(\Sigma', \mathscr{L}', \xi'), F(\Sigma, \mathscr{L}, \xi)) \tag{12}$$

$$(m, n) \mapsto m \tag{13}$$

is a covariant functor.

We can also connect a state property system to a closure space and a morphism to a continuous map.

Theorem 2. The correspondence G: **Cls** $\to$ **SP** consisting of
(1) the mapping

$$|\mathbf{Cls}| \to |\mathbf{SP}| \tag{14}$$

$$(\Sigma, \mathscr{F}) \mapsto G(\Sigma, \mathscr{F}) = (\Sigma, \mathscr{F}, \xi) \tag{15}$$

where $\xi: \Sigma \to \mathscr{P}(\mathscr{F}): p \mapsto \{F \in \mathscr{F} | p \in F\}$.
(2) for every pair of objects $(\Sigma, \mathscr{F})$, $(\Sigma', \mathscr{F}')$ of **Cls** the mapping

$$\mathbf{Cls}((\Sigma', \mathscr{F}'), (\Sigma, \mathscr{F})) \to \mathbf{SP}(G(\Sigma', \mathscr{F}'), G(\Sigma, \mathscr{F})) \tag{16}$$

$$m \mapsto (m, m^{-1}) \tag{17}$$

is a covariant functor.

*Theorem 3 (Equivalence of **SP** and **Cls**).* The functors F: **SP** $\to$ **Cls** and G: **Cls** $\to$ **SP** establish an equivalence of categories.

2. ATOMISTICITY AND T_1 SEPARATION

In refs. 2 and 8 it is proved that T_0 closure spaces correspond to the state property systems $(\Sigma, \mathscr{L}, \xi)$ for which the function $\xi: \Sigma \to \mathscr{P}(\mathscr{L})$ is injective. For these state property systems a state is completely determined by the set of all properties it makes actual, and they are called state-determined state property systems. Faure [5] showed that the category of T_1 closure spaces (and continuous maps) is equivalent to the category of complete atomistic lattices (and morphisms). In this section a similar result in our context is given.

Definition 3. A closure space $(X, \mathscr{F})$ is called T_1 if $\forall x \in X$: x is closed, i.e., $cl(x) = x$.

Definition 4. Let $(\Sigma, \mathscr{L}, \xi)$ be a state property system. Then the map s_ξ maps a state p to the strongest property it makes actual, i.e.,

$$s_\xi: \quad \Sigma \to \mathcal{L}: \quad p \mapsto \wedge\xi(p) \tag{18}$$

Proposition 1. Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system. The following are equivalent:

(1) $\xi: \Sigma \to \mathcal{P}(\mathcal{L})$ is injective and $\forall p \in \Sigma$: $s_\xi(p)$ is an atom of $\mathcal{L}$.

(2) $\forall p, q \in \Sigma: p < q \Rightarrow p = q$.

(3) $F(\Sigma, \mathcal{L}, \xi) = (\Sigma, \kappa(\mathcal{L}))$ is a T_1 closure space.

Proof. (1) $\Rightarrow$ (2). Let $p, q \in \Sigma$ with $p < q$. Then we have $s_\xi(p) < s_\xi(q)$. Since $s_\xi(p)$ and $s_\xi(q)$ are atoms of $\mathcal{L}$, we have that $s_\xi(p) = s_\xi(q)$. Since $\xi: \Sigma \to \mathcal{P}(X)$ is injective, s_ξ is also injective. So we have $p = q$.

(2) $\Rightarrow$ (3). Suppose $q \in cl(p)$. Then we have $q \in \kappa(a)$ for every $p \in \kappa(a)$. This means that $a \in \xi(q)$ for every $a \in \xi(p)$. Consequently, $\xi(p) \subset \xi(q)$. Hence $q < p$. By (2) we have that $p = q$. This shows that $cl(p) = p$, for every $p \in \Sigma$.

(3) $\Rightarrow$ (1). Since a T_1 closure space is T_0, it follows that $\xi: \Sigma \to \mathcal{P}(\mathcal{L})$ is injective. Now we show that $s_\xi(p)$ is an atom of $\mathcal{L}$, for $p \in \Sigma$. Let $a \in \mathcal{L}$ with $a < s_\xi(p)$. Then $\kappa(a) \subset \kappa(s_\xi(p))$. Since $\kappa(s_\xi(p)) = \kappa(\wedge\xi(p)) = \cap\kappa(\xi(p)) = \cap\{\kappa(a) | a \in \xi(p)\} = \cap\{\kappa(a) | p \in \kappa(a)\} = cl(p) = p$, we have that $\kappa(a) = \emptyset$ or $\kappa(a) = \{p\}$. This means that $a = 0$ or $a = s_\xi(p)$. ■

For each state property system $(\Sigma, \mathcal{L}, \xi)$, $\{s_\xi(p) | p \in \Sigma\}$ is an order-generating subset of $\mathcal{L}$ [2]. So if for all $p \in \Sigma$, $s_\xi(p)$ is an atom of $\mathcal{L}$, then the lattice $\mathcal{L}$ is atomistic. Hence, if a state property system satisfies one (and hence all) of the conditions in Proposition 1, then $\mathcal{L}$ is atomistic. Accordingly we call such a state property system 'atomistic'.

Definition 5 (Atomistic state property system). A state property system $(\Sigma, \mathcal{L}, \xi)$ is called an atomistic state property system if $\forall p, q \in \Sigma: p < q \Rightarrow p = q$.

Definition 6. We define $\mathbf{SP}_1$ as the full subcategory of $\mathbf{SP}$ where the objects are the atomistic state property systems. Similarly we use $\mathbf{Cls}_1$ for the category of T_1 closure spaces with continuous maps as morphisms.

Proposition 2. Let $(\Sigma, \mathcal{F})$ be a closure space. Then

$$(\Sigma, \mathcal{F}) \in |\mathbf{Cls}_1| \Leftrightarrow G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \xi) \in |\mathbf{SP}_1| \tag{19}$$

where $\xi: \Sigma \to \mathcal{P}(\mathcal{F}): p \mapsto \{F \in \mathcal{F} | p \in F\}$.

Proof. Let $(\Sigma, \mathcal{F})$ be a T_1 closure space. We will prove that $G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \xi)$ is an atomistic state property system by using Proposition 1. We first show that $s_\xi(p)$ is an atom of $\mathcal{F}$, for $p \in \Sigma$. Let $F \in \mathcal{F}$ and $F \subset s_\xi(p)$. Then we have $F \subset s_\xi(p) = \wedge\xi(p) = \cap\xi(p) = \cap\{F \in \mathcal{F} | p \in F\} = cl(p) = p$. So it follows that $F = \emptyset$ or $F = \{p\} = s_\xi(p)$. This shows that $s_\xi(p)$ is an atom of $\mathcal{F}$. The injectivity of ξ follows immediately from the fact that a T_1 closure space is also T_0. Conversely, if $G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \xi) \in |\mathbf{SP}_1|$, then we have that $\forall p, q \in \Sigma$: $p < q \Rightarrow p = q$. By Proposition 1 we have $(\Sigma, \mathcal{F}) = FG(\Sigma, \mathcal{F}) \in |\mathbf{Cls}_1|$. ■

Theorem 4. The covariant functors F and G restrict and corestrict to functors F: $\mathbf{SP}_1 \to \mathbf{Cls}_1$ and G: $\mathbf{Cls}_1 \to \mathbf{SP}_1$ which establish an equivalence of categories. Hence $\mathbf{SP}_1 \approx \mathbf{Cls}_1$.

Proof. This is an immediate consequence of Theorem 3 and Propositions 1 and 2. ■

3. PRODUCT OF STATE PROPERTY SYSTEMS

In this section I will construct the product of state property systems using the equivalence between the categories **SP** and **Cls**. Since **Cls** is a topological category (over **Set**), the product of a family of closure spaces $(X_i, \mathcal{F}_i)_{i\in I}$ is the closure space $(X, \mathcal{F})$, where $X = \Pi_{i\in I} X_i$ is the Cartesian product of the family $(X_i)_{i\in I}$ and $\mathcal{F}$ is the initial closure structure with respect to $(X, \pi_i, (X_i, \mathcal{F}_i))$. I first recall the construction of the initial closure structure with respect to a given source. For the proof of the following propositions, see refs. 4 and 7.

*Proposition 3 (Initial structures in **Cls**).* Let X be a set, $(X_i, \mathcal{F}_i)_{i\in I}$ a family of closure spaces, and $(f_i: X \to X_i)_{i\in I}$ a family of maps. Then $\mathcal{F} = \{\cap_{i\in I} f_i^{-1}(F_i) | F_i \in \mathcal{F}_i\}$ is the initial closure structure with respect to $(X, f_i, (X_i, \mathcal{F}_i))$.

*Proposition 4 (Products in **Cls**).* Let $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ be closure spaces. Then $((X, \mathcal{F}), (\pi_1, \pi_2))$ is the product of $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ in **Cls**, where

$$X = X_1 \times X_2 \quad \text{(Cartesian product)} \tag{20}$$

$$\pi_i: \quad X \to X_i: \quad (x_1, x_2) \to x_i \tag{21}$$

$$\mathcal{F} = \{\cap_{i\in\{1,2\}} \pi_i^{-1}(F_i) | F_i \in \mathcal{F}_i, i \in \{1,2\}\} \tag{22}$$

$$= \{F_1 \times F_2 | F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\} \tag{23}$$

To simplify the calculations, I will construct the product of two state property systems, which can be generalized to arbitrary families of state property systems.

Theorem 5. Let $(\Sigma_1, \mathcal{L}_1, \xi_1)$ and $(\Sigma_2, \mathcal{L}_2, \xi_2)$ be state property systems. Then $((\Sigma, \mathcal{L}, \xi), ((\pi_1, \pi_1^{-1} \circ \kappa_1), (\pi_2, \pi_2^{-1} \circ \kappa_2)))$ is the product of $(\Sigma_1, \mathcal{L}_1, \xi_1)$ and $(\Sigma_2, \mathcal{L}_2, \xi_2)$, where

$$\Sigma = \Sigma_1 \times \Sigma_2 \tag{24}$$

$$\mathscr{L} = \{\kappa_1(a_1) \times \kappa_2(a_2) | a_i \in \mathscr{L}_i\} \tag{25}$$

$$\kappa_1(a_1) \times \kappa_2(a_2) < \kappa_1(b_1) \times \kappa_2(b_2) \Leftrightarrow a_1 < b_1 \quad \text{and} \quad a_2 < b_2 \tag{26}$$

$$\vee_i(\kappa_1(a_1^i) \times \kappa_2(a_2^i)) = \kappa_1(\vee_i a_1^i) \times \kappa_2(\vee_i a_2^i) \tag{27}$$

$$\wedge_i(\kappa_1(a_1^i) \times \kappa_2(a_2^i)) = \kappa_1(\wedge_i a_1^i) \times \kappa_2(\wedge_i a_2^i) \tag{28}$$

$$\xi: \quad \Sigma \to \mathscr{P}(\mathscr{L}): \quad (p_1, p_2) \mapsto \{\kappa_1(a_1) \times \kappa_2(a_2) | a_i \in \xi_i(p_i)\} \tag{29}$$

$$\pi_i: \quad X \mapsto X_i: \quad (x_1, x_2) \mapsto x_i \tag{30}$$

Proof. Using the equivalence functor F: **SP** $\to$ **Cls** as proposed in Theorem 1, we find the closure spaces $(\Sigma_1, \kappa_1(\mathscr{L}_1))$ and $(\Sigma_2, \kappa_2(\mathscr{L}_2))$. By the previous proposition we find that $((\Sigma, \mathscr{L}), (\pi_1, \pi_2))$ is the product of these two closure spaces, where

$$\Sigma = \Sigma_1 \times \Sigma_2 \tag{31}$$

$$\mathscr{L} = \{\kappa_1 (a_1) \times \kappa_2 (a_2) | a_i \in \mathscr{L}_i\} \tag{32}$$

Since the functor G: **Cls** $\to$ **SP** from Theorem 2 is an equivalence functor, $G((\Sigma, \mathscr{L}), (\pi_1, \pi_2))$ is the product of $G(\Sigma_1, \kappa_1(\mathscr{L}_1))$ and $G(\Sigma_2, \kappa_2(\mathscr{L}_2))$.

We have

$$G(\Sigma, \mathscr{L}) = (\Sigma, \mathscr{L}, \xi) \tag{33}$$

$$\xi: \quad \Sigma \to \mathscr{P}(\mathscr{L}): \quad (p_1, p_2) \mapsto \{\kappa_1(a_1) \times \kappa_2(a_2) | a_i \in \xi_i(p_i)\} \tag{34}$$

$$G(\pi_i) = (\pi_i, \pi_i^{-1}): \quad (\Sigma, \mathscr{L}, \xi) \to G(\Sigma_i, \kappa_i(\mathscr{L}_i)) \tag{35}$$

$(Id_{\Sigma_1}, \kappa_1)$: $G(\Sigma_1, \kappa_1(\mathscr{L}_1)) \to (\Sigma_1, \mathscr{L}_1, \kappa_1)$ and

$$(Id_{\Sigma_2}, \kappa_2): G(\Sigma_2, \kappa_2(\mathscr{L}_2)) \to (\Sigma_2, \mathscr{L}_2, \xi_2)$$

are isomorphisms. Hence,

$$(G(\Sigma, \mathscr{L}), ((Id_{\Sigma_1}, \kappa_1) \circ G(\pi_1), (Id_{\Sigma_2}, \kappa_2) \circ G(\pi_2)))$$
$$= ((\Sigma, \mathscr{L}, \xi), ((\pi_1, \pi_1^{-1} \circ \kappa_1), (\pi_2, \pi_2^{-1} \circ \kappa_2)))$$

is the product of $(\Sigma_1, \mathscr{L}_1, \xi_1)$ and $(\Sigma_2, \mathscr{L}_2, \xi_2)$. ∎

Remark 1. (a) The product presented in ref. 2 is of course isomorphic to the construction in Theorem 5.

(b) $\mathbf{Cls}_1$ is closed under formation of products in **Cls**, i.e., if we consider a family of T_1 closure spaces, then the product formed in **Cls** is also a T_1 closure space. By Theorem 4 it follows that the product of an arbitrary family of atomistic state property systems is an atomistic state property system.

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